Mathematics without formulas?[1]

Willi Dörfler

Introduction

Beyond the trivial observation that signs of many kinds play a central role in all mathematical activities I will give a more far reaching motivation and justification for the *relevance of semiotic theories within mathematics education.* This has to do with a widespread tendency – at least in my view – in school mathematics to marginalize mathematical operations and calculations be they arithmetical, algebraic or geometric. I will first describe the main features of that tendency and then try to point out arguments against it deriving from semiotic positions proposed by Peirce and Wittgenstein. This will also show the relevance of theories and theoretical positions for the practice of mathematical education which without such sound foundations is in danger to go astray and to follow ideological fashions.

Meaning of signs

One central task in teaching mathematics at all levels is to offer the learners the opportunity to develop and construct meaning and understanding for the respective mathematics and especially for the signs and symbols used therein. Now there is the orthodox and classic view that signs gain their meaning from the objects they stand for, which they designate. From this designation and reference also the rules for the manipulation of the signs and symbols derive and are thereby also justified. For school mathematics this conviction leads to two different ways of methodological approaches to develop meaning in mathematics education. One is of an empiricist character and the other of an idealistic one. Thereby I do not assert that all of mathematics teaching falls into one of these two categories or a combination of them. Yet, I see a strong emphasis in school on basic orientations which can be subsumed under these labels and which is to the detriment of what one could call "formal" mathematics.

Empiricist foundation

The empiricist orientation intends to convey meaning in mathematics via what one can call every day applications. The use of mathematics for describing practical and non-mathematical situations and processes and for solving the respective problems is the focus of teaching and learning and foremost of the tasks presented to the learners. In a way, one wants to import meaning into mathematics from outside of mathematics thereby possibly confounding meaning and relevance. Without those references to concrete objects the signs and symbols of mathematics (like the numerals of arithmetic or the letters of algebra) and the operations with them are said to be meaningless.

Looking at the tasks posed one can realize some other features characteristic for this approach. In many of them very little to none calculations are needed for solving the task or those can be delegated to a computer. The emphasis in the tasks is on the one hand on the process of model building and on the other hand on a variety of completely non-mathematical questions. These are concerned with economical, ecological, social or humanistic aspects or with questions from the respective context (like say biology, physics, technology, etc.). The students are urged to discuss those aspects but mostly in a way divorced from mathematics or not in need of mathematics. For sure, there are also discussions about the role played by mathematics and about the appropriateness of mathematical modelling. But for all that usually little knowledge of the mathematics is a prerequisite and calculations mostly can be avoided more or less completely.

Also more comprehensive projects are staged in the same vein which necessitate cooperation of several students. All that is not bad in itself and of course much is to be learned in this way, yet very little about mathematics itself, its notions and operations. One must also take into account the strict time constraints of school teaching to see that such an approach with a lot of discussion and group work leaves little room for anything else. *Mathematics so to say is dissolved into its applications.*

Discussing modelling

The situation is in my view even exacerbated by the phenomenon that there can be observed a further shift beyond the one described so far (from doing mathematics to using and applying it): many

[1] Manuskript eines im Rahmen des Thematic Afternoon auf dem ICME-13 in Hamburg gehaltenen Vortrags.

tasks demand not just to carry out and justify a modelling process but to focus on discussing that process itself. That again is a shift from doing to discussing or to simply talking. One even finds test items of this quality. This goes under headings like: awareness, responsibility, consciousness, ability for rational judgment. There is apparently the pedagogical conviction that one can sensibly talk about mathematics and its ramifications without having experiences with doing mathematics. And one finds the outspoken opinion that the students in school should not learn mathematics as such and its operations but learn rather about mathematics, its uses in society and the appropriateness thereof.

Ideas instead of calculations

This shift from mathematics proper to the meta-level of talk about mathematics is also significant with the second tendency which I will discuss now rather shortly. It shows itself in notions like "fundamental" ideas among which one finds: number, measure, approximation, linearity, probability, function. And similar to what we found in the empiricist context again here is a strong negligence and even disregard for the role to be played by the various mathematical sign systems and notations. The mathematical signs and symbols function only to express ideas and the latter come first, it is said; and the situation is compared to music composition where purportedly the music comes first and the score only in the hindsight denotes and communicates the former. The students must therefore be acquainted with the ideas first which can only be done in a rather loose and imprecise way.

The mathematical notations are downplayed as secondary. To understand the "big" ideas no routine or experience with mathematical operations is needed, it is assumed, and the ideas as such convey a deeper understanding of mathematics as it is possible to be attained by carrying out typical mathematical operations like solving equations, calculating an integral or devising a proof.

Truncation of mathematics

Both positions could be seen as avoiding or marginalizing the use of and the manipulation of mathematical formulas possibly in a trial to make mathematics more palatable and less frightening. But, the "formula" and its formal uses is one of the great inventions of mathematics. Thus those tendencies lead to a far reaching truncation of mathematics in the school. That this really is a kind of threat is borne out by the views of two very prominent philosophers who have devoted much of their thinking to mathematics and mathematical reasoning.

Charles Sanders Peirce

I will start with the American philosopher, logician and mathematician Charles Sanders Peirce (1839-1914) and his notions of *diagram and diagrammatical reasoning*. Of course what follows is only a very rough sketch! In Peirce diagrams are among others arithmetic and algebraic terms and formulas, equations, geometric figures, graphics, formulas of all kind, for short all relevant mathematical inscriptions or sign systems. And for all systems of (mathematical) diagrams there are rules for their manipulation which is a general form of calculation. The main assertion by Peirce now is that mathematical thinking of all kinds from simple calculations to complex proofs essentially consists in the manipulation and transformation and in the invention of diagrams which latter thereby do not express anything which is outside of the diagrams. In a pointed way Peirce says that mathematical reasoning occurs on the paper where the diagrams are written. Convincing examples show that this diagrammatic reasoning is wholly self-contained and that it lends to mathematics a great autonomy based on the structure of the diagrams and the respective operation rules. *Understanding mathematics then can and even must be equated with diagrammatic experience and fluency in diagrammatic reasoning which to a great extent is based on formulas including those of a geometric character.*

Here one realizes the gulf between the views of Peirce and the tendencies sketched above! This becomes very transparent when one looks at a draft by Peirce for a text book of elementary arithmetic which bases the learning of it on diagrammatic activities like ways of moving forth and back in the number sequence. Peirce also says that mathematical development in general and in the individual as well is constituted by the invention, construction or acquisition of systems of diagrams. To learn mathematics depends on learning diagrams and the operations with them. This also holds for the applications of mathematics where diagrams are used as signs (i.e. models) for structures outside of mathematics.

Ludwig Wittgenstein

The Austrian philosopher Ludwig Wittgenstein (1889–1951) even was more radical with regard to the central role of signs for mathematics and mathematical activities. Thereby he is concerned with mathematics proper or "pure" mathematics. Yet he says that mathematics derives importance and relevance from its every day applications but does not receive meaning from outside. *The meaning of the mathematical signs and symbols resides in their use* *within mathematics, i.e. in the manifold operations with them in calculations of all sorts and especially in proofs.* For making this view more transparent Wittgenstein has proposed the notion of a language game or more specifically for mathematics that of a sign game. In a game like chess the meaning of the figures is determined by the rules of the game and likewise the meaning of the mathematical symbols is tantamount to the operation rules and all of their consequences. Thus he says that in a mathematical argument one cannot appeal to the meaning of the terms since this meaning is only developed within mathematics. A simple conclusion would then be that learning mathematics consists in genuinely and progressively taking part in the respective sign games. More specifically, learning arithmetic means learning to calculate and solve arithmetic problems of all sort. Likewise this holds for algebra and any other part of (school) mathematics. To understand what a mathematical term means is equated with the ability to use it in a correct way, i.e. according to the respective rules for operation.

This is a strong argument for a kind of mathematical training which possibly is analogous to the training necessary for becoming a fairly good chess player: to understand chess simply consists in being able to play it well. Wittgenstein also proposes the view that the signs of mathematics do not designate given objects like numbers from which their meaning derives. Here again the analogy with chess might help: a figure of chess, say the queen, does not designate anything. Thus fundamental ideas do not by themselves elucidate the mathematical use of signs and possibly it is the other way round that the big ideas can only be understood from their instantiations within mathematics by formal models e.g. of linearity. Of course do have many mathematical sign games their genetic roots in practical problems. But they cannot be reduced to them and they always essentially transcend them. This becomes very clear already in the case of negative and of rational numbers.

Summarizing

To sum up: the positions taken by Peirce and Wittgenstein strongly oppose the current tendency to reduce or to neglect the role of symbolic mathematical activity. Even if the goal of school mathematics is rather the applications and a more general and superficial knowledge about some features of mathematics this goal will only be achievable when a genuine understanding of the mathematics involved is developed. And such an understanding according to Peirce and Wittgenstein presupposes familiarity with the mathematical diagrams and/or with the mathematical sign games. *In other words, there is no sensible mathematics without formulas.*

Willi Dörfler, Alpen-Adria-Universität Klagenfurt
Email: willi.doerfler@aau.at

„Digitale Bildung" — ein Bildungskonzept?

Horst Hischer

Kein Mensch lernt digital. Es gibt weder digitalen Unterricht noch digitale Bildung [. . .].
Ralf Lankau, 2017

Ausgangslage

„Digitalisierung" ist in aller Munde. So überboten sich in den Wochen vor der letzten Bundestagswahl viele Parteien mit oft dubios bleibenden Parolen zur Forcierung einer vorgeblich notwendigen „Digitalisierung", als deren – höflich formuliert: – bedenklichste hier *„Digital first. Bedenken second."* genannt sei.

Aber auch im Kontext von Bildung, Schule, Bildungspolitik und Didaktik finden wir – in den letzten Jahren zunehmend, nun auch bis in die tagespolitische Berichterstattung von Presse und Fernsehen hinein – Fokussierungen auf „Digitalisierung", die nun sogar Forderungen z. B. nach „Digitaler Bildung" und „Digitalem Lernen" nach sich ziehen. Und die *Gesellschaft für Didaktik der Mathematik* (GDM) hat im Juli 2017 ein Positionspapier veröffentlicht, bei dem es expressis verbis u. a. auch um „digitale Bildung" geht. [1]

Aber können denn „Bildung" und „Lernen" digital sein? Wurde das bei der Wortwahl sorgsam

[1] ojs.didaktik-der-mathematik.de/index.php/mgdm/article/view/59/205